On the Ball-Marsden-Slemrod obstruction for bilinear control systems

Nabile Boussaid, Marco Caponigro, Thomas Chambrion

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Nabile Boussaïd
Laboratoire de Mathématiques de Besançon, UMR 6623
Université de Bourgogne Franche-Comté, Besançon, France
Nabile.Boussaid@univ-fcomte.fr
Marco Caponigro
Équipe M2N
Conservatoire National des Arts et Métiers, Paris, France
Marco.Caponigro@cnam.fr
Thomas Chambrion
Universite de Lorraine, CNRS, Inria, IECL, Nancy, France
Thomas.Chambrion@univ-lorraine.fr

***Abstract*— In this paper we present an extension to the case of L^1-controls of a famous result by Ball–Marsden–Slemrod on the obstruction to the controllability of bilinear control systems in infinite dimensional spaces.**

I. INTRODUCTION

A. Bilinear control systems

Let X be a Banach space, $A : D(A) \to X$ a linear operator in X with domain $D(A)$, $B : X \to X$ a linear bounded operator and ψ_0 an element in X.

We consider a following bilinear control system on X

$$\begin{cases} \dot{\psi}(t) &= A\psi(t) + u(t)B\psi(t), \\ \psi(0) &= \psi_0, \end{cases} \tag{1}$$

where $u : [0, +\infty) \to \mathbf{R}$ is a scalar function representing the control.

Assumption 1: The pair (A, B) of linear operators in X satisfies

1) the operator A generates a C^0-semigroup of linear bounded operators on X.
2) the operator B is bounded.

Definition 1: Let (A, B) satisfy Assumption 1. A function $\psi : [0, +\infty) \to X$ is a *mild solution* of (1) if ψ is continuous from $[0, +\infty)$ to X and, for every t in $[0, T]$,

$$\psi(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}B\psi(s)u(s)\mathrm{d}s \tag{2}$$

Equation (2) is often called Duhamel formula. Existence and uniqueness of mild solutions for equation (1) is given by the following result (see, for instance, Proposition 2.1 and Remark 2.7 in [BMS82]).

Proposition 1: Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X, for every u in $L^1_{loc}([0, +\infty), \mathbf{R})$, there exists a unique mild solution $t \mapsto \Upsilon^u_{t,0}\psi_0$ to the Cauchy problem (1). Moreover, for every ψ_0 in X, the end-point mapping $\Upsilon_{\cdot,0}\psi_0 : [0, +\infty) \times L^1_{loc}([0, +\infty), \mathbf{R}) \to X$ is continuous.

For the sake of completeness, we will prove a weak version of Proposition 1 (without the continuity statement), sufficient for our purpose, in Section III-C.

Definition 2: Assume that (A, B) satisfies Assumption 1 and let $\mathcal{U}$ be a subset of $L^1_{loc}([0, +\infty), \mathbf{R})$. For every ψ_0 in X, the *attainable set from ψ_0 with controls in $\mathcal{U}$* is defined as

$$\mathcal{A}(\psi_0, \mathcal{U}) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{U}} \{\Upsilon^u_{T,0}\psi_0\}.$$

Our main result is the following property of the attainable set of system (1) with L^1 controls.

Theorem 2: Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X, the attainable set $\mathcal{A}(\psi_0, L^1_{loc}([0, +\infty), \mathbf{R}))$ from ψ_0 with L^1_{loc} controls is contained in a countable union of compact subsets of X.

B. The Ball–Marsden–Slemrod obstruction

Our main result, Theorem 2 is an extension of the well-known Ball–Marsden–Slemrod obstruction to controllability (see also [ILT06]) which is as follows.

Theorem 3 (Theorem 3.6 in [BMS82]): Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X, the attainable set $\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbf{R}))$ from ψ_0 with L^r_{loc} controls, $r > 1$, is contained in a countable union of compact subsets of X.

A consequence of Theorem 3 to the framework of the conservative bilinear Schrödinger equation is given by Turinici.

Theorem 4 (Theorem 1 in [Tur00]): Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X, the set $\cup_{\alpha>0} \alpha\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbf{R}))$ is contained in a countable union of compact subsets of X.

Theorems 2 and 3 are basically empty in the case in which X is finite dimensional, since, in this case, X itself is a countable union of compact sets. On the other hand, when X is infinite dimensional, these results represent a strong topological obstruction to the exact controllability. Indeed, compact subsets of an infinite dimensional Banach space have empty interiors and so is a countable union of closed subsets with empty interiors (as a consequence of Baire Theorem).

Whether the non-controllability result of Ball, Marsden, and Slemrod, Theorem 3 holds for L^1 control has been an open question for decades. Indeed, the proof of Theorem 3 does not apply directly to the L^1 case. To see what fails let us briefly recall the method used in [BMS82] for the proof of Theorem 3. The first step is to write

$$\begin{aligned}
&\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0,+\infty), \mathbf{R})) \\
&\quad = \bigcup_{T\geq 0} \bigcup_{r>1} \bigcup_{u \in L^r([0,T],\mathbf{R})} \{\Upsilon^u_{T,0}\psi_0\} \\
&\quad = \bigcup_{l\in\mathbf{N}} \bigcup_{m\in\mathbf{N}} \bigcup_{k\in\mathbf{N}} \bigcup_{0\leq t\leq l} \left(\bigcup_{\|u\|_{L^{1+1/m}}\leq k} \{\Upsilon^u_{t,0}\psi_0\} \right).
\end{aligned}$$

Hence it is sufficient to prove that, for every (l,m,k) in $\mathbf{N}^3$, the set

$$\mathcal{A}^{l,m,k} = \bigcup_{0\leq t\leq l} \left(\bigcup_{\|u\|_{L^{1+1/m}}\leq k} \{\Upsilon^u_{t,0}\psi_0\} \right)$$

has compact closure in X. To this end, one considers a sequence $(\psi_n)_{n\in\mathbf{N}}$ in $\mathcal{A}^{l,m,k}$, associated with a sequence of times $(t_n)_{n\in\mathbf{N}}$ in $[0,l]$ and a sequence of controls $(u_n)_{n\in\mathbf{N}}$ in the ball of radius k of $L^{1+1/m}([0,+\infty),\mathbf{R})$. By compactness of $[0,l]$, up to extraction, one can assume that $(t_n)_{n\in\mathbf{N}}$ tends to t_∞ in $[0,l]$. By Banach–Alaoglu–Bourbaki Theorem, the balls of $L^{1+1/m}([0,+\infty),\mathbf{R})$ are weakly (sequentially) compact and, hence, up to extraction, one can assume that $(u_n)_{n\in\mathbf{N}}$ converges weakly in $L^{1+1/m}([0,+\infty),\mathbf{R})$ to some u_∞. The hard step of the proof (Lemma 3.7 in [BMS82]) is then to show that $\Upsilon^{u_n}_{t_n,0}\psi_0$ tends to $\Upsilon^{u_\infty}_{t_\infty,0}\psi_0$ as n tends to infinity.

A crucial point in the proof of Theorem 3 given in [BMS82] is the fact that the closed balls of L^p, $p>1$ are weakly sequentially compact. This is no longer true for the balls of L^1, and this prevents a direct extension of the proof of Theorem 3 to Theorem 2. Here we present a brief and self-contained proof of Theorem 2 mainly based on Dyson expansions and basics compacteness properties on Banach spaces.

An alternative proof of Theorem 3, not relying on the reflectiveness of the set of admissible controls, has been recently given in [BCC17]. The proof applies to a very large class of controls (namely, Radon measures) which contains locally integrable functions, and for its generality it is technically quite involved, in contrast with the simplicity of the underlying ideas. It applies also, with minor modifications, to nonlinear problems [CT19].

C. Content

In this note we present a simple proof of Theorem 2. However, historical reasons have made different communities use incompatible terminologies and, in order to avoid ambiguities, we present in Section II a quick reminder of basic facts in Banach topologies. Section III gives a short introduction to the classical Dyson expansion (Section III-A), the proof of an instrumental compactness property (Section III-B) and a proof of the well-posedness of the Cauchy problem (1) (Section III-C). We conclude in Section IV with the proof of Theorem 2.

II. Basic Facts About Topology in Banach Spaces

A. Notations

The Banach space X is endowed with norm $\|\cdot\|$. For every ψ_c in X and every $r>0$, $B_X(\psi_c,r)$ denotes the ball of center ψ_c and of radius r:

$$B_X(\psi_c,r) = \{\psi \in X | \|\psi - \psi_c\| < r\}.$$

In the following, all we need to know about generators of C^0-semigroup is the classical result stated in Proposition 5 (see Chapter VII of [HP57]).

Proposition 5: Assume that A generates a C^0-semigroup. Then there exist $M,\omega>0$ such that $\|e^{At}\| \leq Me^{\omega t}$ for every $t\geq 0$.

B. Compact subset of Banach spaces

Definition 3: Let X be a Banach space and Y be a subset of X. A family $(O_i)_{i\in I}$, indexed by an arbitrary set I, is an open cover of Y if O_i is open in X for every i in I and $Y \subset \cup_{i\in I} O_i$.

Definition 4: Let X be a Banach space. A subset Y of X is said to be *compact* if from any open cover of Y, it is possible to extract a finite cover of Y.

Definition 5: Let X be a Banach space. A subset Y of X is said to be *sequentially compact* if from any sequence $(\psi_n)_{n\in\mathbf{N}}$ taking value in Y , it is possible to extract a subsequence $(\psi_{\phi(n)})_{n\in\mathbf{N}}$ converging in Y.

Definition 6: Let X be a Banach space. A subset Y of X is said to be *totally bounded* if for every $\varepsilon>0$, there exist $N\in\mathbf{N}$ and a finite family $(x_i)_{1\leq i\leq N}$ in X such that

$$Y \subset \bigcup_{i=1}^N B_X(x_i,\varepsilon).$$

Proposition 6: Let X be a Banach space. For every subset Y of X, the following assertions are equivalent:

1) Y is compact.
2) Y is sequentially compact.
3) Y is complete and totally bounded.
4) Y is closed and totally bounded.

Proposition 7: Let X be a Banach space, N in $\mathbf{N}$ and $(Y_i)_{1\leq i\leq N}$ be a finite family of compact subsets of X. Then, the finite sum

$$\sum_{i=1}^N Y_i = \{y_1+y_2+\ldots+y_N \mid y_i \in Y_i, i=1,\ldots,N\}$$

is compact as well.

Proposition 8: Let X be a Banach space, N in $\mathbf{N}$ and $(Y_i)_{1\leq i\leq N}$ be a finite family of totally bounded subsets of X. Then, the finite sum

$$\sum_{i=1}^N Y_i = \{y_1+y_2+\ldots+y_N \mid y_i \in Y_i, i=1,\ldots,N\}$$

is totally bounded as well.

Proposition 9: Let X be a Banach space, $T > 0$ and (A, B) satisfy Assumption 1. Define the mapping

$$\begin{array}{rccc} F: & [0,T]\times[0,T]\times X & \to & X \\ & (s,t,\psi) & \mapsto & e^{|t-s|A}B\psi \end{array}$$

Then, for every totally bounded subset Y of X, the set $F([0,T]\times[0,T]\times Y)$ is totally bounded as well.

Proof: We claim that $G : (t, \psi) \mapsto e^{tA}\psi$ is jointly continuous in its two variables. Indeed, for every ψ, ψ_0 in X, for every $t, t_0 \geq 0$,

$$\begin{aligned} \|e^{tA}\psi - e^{t_0 A}\psi_0\| &\leq \|e^{tA}(\psi-\psi_0)\| + \|(e^{tA}-e^{t_0A})\psi_0\| \\ &\leq Me^{\omega t}\|\psi-\psi_0\| + \|(e^{tA}-e^{t_0A})\psi_0\|. \end{aligned}$$

This last quantity tends to zero as (t, ψ) tends to (t_0, ψ_0). As a consequence, F is continuous (as composition of continuous functions).

If Y is totally bounded, the topological closure $\bar{Y}$ of Y is compact (because the ambient space X is complete). Hence $[0,T]\times[0,T]\times\bar{Y}$ is compact. By continuity, $F([0,T]\times[0,T]\times\bar{Y})$ is compact, hence is totally bounded. The set $F([0,T]\times[0,T]\times Y)$, which is contained in $F([0,T]\times[0,T]\times\bar{Y})$, is, therefore, totally bounded as well. ∎

C. Partition of unity in Banach spaces

Definition 7: Let X be a Banach space. A family $(x_i)_{i\in I}$ of points of X is *locally finite* if for every x in X and every $R > 0$, the cardinality of the set

$$\left(\bigcup_{i\in I}\{x_i\}\right)\cap B_X(x,R)$$

is finite.

Definition 8: Let X be a Banach space, Y be a subset of X, and $(O_i)_{i\in I}$ be an open cover of Y. A family $(\phi_i)_{i\in I}$ of continuous functions from Y to $[0,1]$ is called a *partition of the unity of* Y *adapted to the cover* $(O_i)_{i\in I}$ if

(i) for every $i \in I$, $\phi_i(x) = 0$ for every $x \notin O_i$;
(ii) $\sum_{i\in I}\phi_i(x) = 1$ for every $x \in Y$.

Proposition 10: Let X be a Banach space, Y be a subset of X, $\delta > 0$, $(x_j)_{j\in J}$ be a locally finite family of points in Y such that $Y \subset \cup_{j\in J}B_X(x_j,\delta)$. Then, there exists a partition of the unity, $(\phi_j)_{j\in J}$, adapted to the open cover $(B(x_j, 2\delta))_{j\in J}$ of Y.

Moreover, if a family $(\phi_j)_{j\in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j,2\delta))_{j\in J}$, then for every x in Y, $\|x - \sum_{j\in J}\phi_j(x)x_j\| \leq 2\delta$.

Proof: We first prove the existence of a partition of the unity adapted to the open covering $(B_X(x_j,2\delta))_{j\in J}$ of Y. To this end, we define, for every j in J, the continuous functions $\varphi_j : X \to [0,1]$ by

$$\begin{cases} \varphi_j(x) = 1, & \text{if } \|x-x_j\| < \delta, \\ \varphi_j(x) = 2 - \|x-x_j\|/\delta, & \text{if } \delta \leq \|x-x_j\| < 2\delta, \\ \varphi_j(x) = 0, & \text{if } 2\delta \leq \|x-x_j\|. \end{cases}$$

Since the family $(x_j)_{j\in J}$ is locally finite, the sum $\sum_{j\in J}\varphi_j(x)$ converges for every x in Y. Moreover, since $Y \subset \cup_{j\in J}B(x_j,\delta)$, the function $x \mapsto \sum_{j\in J}\varphi_j(x)$ does not vanish on Y. For every j_0 in J, we define ϕ_{j_0} by

$$\phi_{j_0}(x) = \varphi_{j_0}(x)\frac{1}{\sum_{j\in J}\varphi_j(x)},$$

and the family $(\phi_j)_{j\in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j,2\delta))_{j\in J}$ of Y.

We now prove the second point of Proposition 10. Let $(\phi_j)_{j\in J}$ be a partition of unity of Y adapted to the cover $(B_X(x_j,2\delta))_{j\in J}$. Then, for every x in Y,

$$\begin{aligned} \left\|x - \sum_{j\in J}\phi_j(x)x_j\right\| &= \left\|\sum_{j\in J}\phi_j(x)x - \sum_{j\in J}\phi_j(x)x_j\right\| \\ &= \left\|\sum_{j\in J}\phi_j(x)(x-x_j)\right\| \\ &\leq \sum_{j\in J}\phi_j(x)\left\|x-x_j\right\|. \end{aligned}$$

By construction, $\phi_j(x) = 0$ as soon as $\|x-x_j\| \geq 2\delta$. Hence,

$$\left\|x - \sum_{j\in J}\phi_j(x)x_j\right\| \leq 2\delta\sum_{j\in J}\phi_j(x) \leq 2\delta,$$

which concludes the proof. ∎

III. DYSON EXPANSION

A. The Dyson Operators

Let (A, B) satisfy Assumption 1. For every u in $L^1_{loc}([0,+\infty),\mathbf{R})$, $p \in \mathbf{N}$, and $t \geq 0$ we define the linear bounded operator $W_p(t,u) : X \to X$ recursively by

$$W_0(t,u)\psi = e^{tA}\psi$$

$$W_p(t,u)\psi = \int_0^t e^{(t-s)A}BW_{p-1}(s,u)\psi u(s)\mathrm{d}s, \quad \text{for } p \geq 1,$$

for every ψ in X. We have the following estimate for the norm of the operator.

Proposition 11: For every u in $L^1_{loc}([0,+\infty),\mathbf{R})$, $p \in \mathbf{N}$, and $t \geq 0$

$$\|W_p(t,u)\| \leq \frac{M^{p+1}e^{(p+1)\omega t}\|B\|^p(\int_0^t|u(s)|\mathrm{d}s)^p}{p!}.$$

Proof: We prove the result by induction on p in $\mathbf{N}$. For $p = 0$ the result clearly follows from Proposition 5. Assume that the result holds for $p \geq 0$. Then, for every ψ in X,

$$\begin{aligned} &\|W_{p+1}(t,u)\psi\| \\ &\quad\leq \left\|\int_0^t e^{(t-s)A}BW_p(s,u)\psi u(s)\mathrm{d}s\right\| \\ &\quad\leq \int_0^t\left\|e^{(t-s)A}\right\|\|B\|\,\|W_p(s,u)\|\,\|\psi\|\,|u(s)|\mathrm{d}s \\ &\quad\leq M\int_0^t e^{(t-s)\omega}\|B\|\frac{M^{p+1}e^{(p+1)\omega s}\|B\|^p(\int_0^s|u(\tau)|\mathrm{d}\tau)^p}{p!} \\ &\qquad\qquad \|\psi\|\,|u(s)|\mathrm{d}s \\ &\quad\leq M^{p+2}e^{(p+2)\omega t}\|B\|^{p+1}\frac{(\int_0^t|u(\tau)|\mathrm{d}\tau)^{p+1}}{(p+1)!}\left\|\psi\right\|. \end{aligned}$$

The third inequality follows from Proposition 5 and the induction assumption at rank p. We conclude the proof by induction on p. ■

B. A compactness property

Lemma 12: For every j in $\mathbf{N}$, $T \geq 0$ and $K \geq 0$, and ψ_0 in X the set

$$\mathcal{W}_j^{T,K} = \{W_j(t,u)\psi_0 \mid 0 \leq t \leq T, \|u\|_{L^1} \leq K\}, \qquad (3)$$

is totally bounded

Proof: We prove the result by induction on j in $\mathbf{N}$. For $j = 0$, consider $\mathcal{W}_0^{T,K} = \{e^{tA}\psi_0, 0 \leq t \leq T\}$ and let $(w_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{W}_0^{T,K}$. Then there exists a sequence $(t_n)_{n\in\mathbf{N}}$ such that $w_n = e^{t_n A}\psi$ for every n. Up to extraction, $\lim_{n\to\infty} t_n = t \in [0,T]$, since $[0,T]$ is compact. By definition of C^0-semigroup, $\lim_{n\to\infty} e^{t_n A}\psi_0 = e^{tA}\psi_0$. This proves that $\mathcal{W}_0^{T,K}$ is sequentially compact, hence compact and, in particular, totally bounded (Proposition 6).

Assume that, for $j \geq 0$, $\mathcal{W}_j^{T,K}$ is totally bounded. By Proposition 9, the set

$$\begin{aligned} Z_j^{T,K} &:= \{e^{(t-s)A}B\psi_0, \psi \in \mathcal{W}_j^{T,K}, 0 \leq s \leq t \leq T\} \\ &\subset F([0,T]^2 \times \mathcal{W}_j^{T,K}) \end{aligned}$$

is totally bounded as well.

Let $\varepsilon > 0$ be given and define $\delta = \frac{\varepsilon}{2K+1} > 0$. Since $Z_j^{T,K}$ is totally bounded, there exists a finite family $(x_i)_{1\leq i\leq N_\delta}$ in $Z_j^{T,K}$ such that

$$Z_j^{T,K} \subset \bigcup_{i=1}^{N_\delta} B_X(x_i,\delta).$$

Let $(\phi_i)_{\leq i\leq N_\delta}$ be a partition of the unity adapted to the cover $\cup_{i=1}^{N_\delta} B(x_i, 2\delta)$ of $Z_j^{T,K}$. Such a partition of the unity exists by Proposition 10, and moreover, for every x in $Z_j^{T,K}$, we have

$$\left\| x - \sum_{i=1}^{N_\delta} \phi_i(x)x_i \right\| \leq 2\delta. \qquad (4)$$

Applying the inequality (4) with $x = e^{(t-s)A}BW_j(s,u)\psi_0$, we get, for every u in $L^1_{loc}([0,+\infty,\mathbf{R})$ and every (s,t) such that $0 \leq s \leq t$,

$$\left\| e^{(t-s)A}BW_j(s,u)\psi_0 - \sum_{i=1}^{N_\delta} \phi_i(e^{(t-s)A}BW_j(s,u)\psi_0)x_i \right\| \leq 2\delta.$$

Multiplying by $u(s)$ and integrating for s in $[0,t]$, one gets, for every u such that $\int_0^t |u| \leq K$,

$$\left\| \int_0^t e^{(t-s)A}BW_j(s,u)\psi_0 u(s)\mathrm{d}s - \sum_{i=1}^{N_\delta} \int_0^t \phi_i(e^{(t-s)A}BW_j(s,u)\psi_0)u(s)\mathrm{d}s\, x_i \right\| \leq 2\delta \int_0^t |u(s)|\mathrm{d}s,$$

that is

$$\left\| \int_0^t e^{(t-s)A}BW_j(s,u)\psi_0 u(s)\mathrm{d}s - \sum_{i=1}^{N_\delta} \int_0^t \phi_i(e^{(t-s)A}BW_j(s,u)\psi_0)u(s)\mathrm{d}s\, x_i \right\| \leq 2\delta K. \qquad (5)$$

The set $\sum_{i=1}^{N_\delta}[0,K]x_i$ is compact by Proposition 7 and, hence, totally bounded. Then, there exists a finite family $(y_i)_{1\leq i\leq N'_\delta}$ such that

$$\sum_{i=1}^{N_\delta}[0,K]x_i \subset \bigcup_{i=1}^{N'_\delta} B_X(y_i,\delta). \qquad (6)$$

From (5) and (6), one deduces that

$$\mathcal{W}_{j+1}^{T,K} \subset \bigcup_{i=1}^{N'_\delta} B_X(y_i,(2K+1)\delta) = \bigcup_{i=1}^{N'_\delta} B_X(y_i,\varepsilon).$$

This proves that $\mathcal{W}_{j+1}^{T,K}$ is totally bounded and concludes the proof. ■

C. Convergence of the Dyson expansion

Let $T > 0$, ψ_0 in X and u in $L^1([0,T],\mathbf{R})$ be fixed. We define $\mathcal{C}([0,T],X)$ the set of continuous functions from $[0,T]$ to X. When endowed with the norm

$$\|\psi\|_{\mathcal{C}([0,T],X)} = \sup_{t\in[0,T]} \|\psi(t)\|,$$

the set $\mathcal{C}([0,T],X)$ turns into a Banach space. We define also the mapping

$$H_T : \mathcal{C}([0,T],X) \to \mathcal{C}([0,T],X)$$

by

$$H_T(\psi)(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}u(s)B\psi(s)\mathrm{d}s$$

for every t in $[0,T]$ and ψ in $\mathcal{C}([0,T],X)$.

Proposition 13: A function ψ in $\mathcal{C}([0,+\infty),X)$ is a mild solution of (1) if and only if, for every $T > 0$, the restriction of ψ to $[0,T]$ is a fixed point of H_T.

Proof: This is a rewriting of the Definition 1. ■

Proposition 14: For every k in $\mathbf{N}$, H_T^k is

$$M^k e^{kT\omega} \frac{\|B\|^k \left(\int_0^T |u(s)|\mathrm{d}s\right)^k}{k!}$$

Lipschitz continuous from $\mathcal{C}([0,T],X)$ to itself.

Proof: Let ψ_1,ψ_2 in $\mathcal{C}([0,T],X)$. We prove by induction on k that, for every k in $\mathbf{N}$, for every t in $[0,T]$,

$$\|H_T^k(\psi_1)(t) - H_T^k(\psi_2)(t)\| \leq M^k e^{kt\omega} \frac{\|B\|^k}{k!} \left(\int_0^t |u(s)|\mathrm{d}s\right)^k \sup_{0\leq s\leq t} \|\psi_1(s) - \psi_2(s)\| \qquad (7)$$

For the step $k=1$, we write, for every t in $[0,T]$,

$$\begin{aligned}
&\|H_T(\psi_1)(t)-H_T(\psi_2)(t)\| \\
&\quad= \left\|\int_{s=0}^{t} e^{(t-s)A}u(s)B(\psi_1(s)-\psi_2(s))\mathrm{d}s\right\| \\
&\quad\leq \int_{s=0}^{t}\left\|e^{(t-s)A}u(s)B(\psi_1(s)-\psi_2(s))\right\|\mathrm{d}s \\
&\quad\leq Me^{t\omega}\|B\|\left(\int_0^t |u(s)|\mathrm{d}s\right)\sup_{0\leq s\leq t}\|\psi_1(s)-\psi_2(s)\|
\end{aligned}$$

and (7) is proved for $k=1$.

Assume that (7) is proved for some k in $\mathbf{N}$. Then, for every t in $[0,T]$,

$$\begin{aligned}
&\|H_T^{k+1}(\psi_1)(t)-H_T^{k+1}(\psi_2)(t)\| \\
&\quad= \left\|\int_{s=0}^{t} e^{(t-s)A}u(s)BH_T^k(\psi_1(s)-\psi_2(s))\mathrm{d}s\right\| \\
&\quad\leq \int_{s=0}^{t}\left\|e^{(t-s)A}u(s)BH_T^k(\psi_1(s)-\psi_2(s))\right\|\mathrm{d}s \\
&\quad\leq Me^{t\omega}\|B\|\int_0^t |u(s)|\sup_{0\leq\tau\leq s}\|H_T^k(\psi_1(\tau)-\psi_2(\tau))\|\mathrm{d}s \\
&\quad\leq Me^{t\omega}\|B\|\int_0^t |u(s)|M^k e^{ks\omega}\left(\int_0^s|u(\tau)|\mathrm{d}\tau\right)^k \\
&\qquad\qquad \frac{1}{k!}\|B\|^k\sup_{0\leq\tau\leq s}\|\psi_1(\tau)-\psi_2(\tau)\|\mathrm{d}s \\
&\quad\leq M^{k+1}e^{(k+1)t\omega}\frac{\|B\|^{k+1}}{(k+1)!}\left(\int_0^t|u(s)|\mathrm{d}s\right)^{k+1} \\
&\qquad\qquad \sup_{0\leq\tau\leq t}\|\psi_1(\tau)-\psi_2(\tau)\|.
\end{aligned}$$

This finishes the proof of (7) for $k+1$. Proposition 14 follows by noting that $\sup_{0\leq\tau\leq t}\|\psi_1(\tau)-\psi_2(\tau)\|\leq\|\psi_1-\psi_2\|_{\mathcal{C}([0,T],X)}$. ∎

Proposition 15: For every u in $L^1_{loc}([0,+\infty),\mathbf{R})$, $T\geq 0$, ψ_0 in X and p in $\mathbf{N}\cap\{0\}$,

$$H_T\left(t\mapsto\sum_{j=0}^{p}W_j(t,0)\psi_0\right)=\left(t\mapsto\sum_{j=0}^{p+1}W_j(t,0)\psi_0\right).$$

Proof: The proof goes by induction on p.

For $p=1$, we write, for every t in $[0,T]$,

$$\begin{aligned}
&H_T(t\mapsto e^{tA})(t) \\
&\quad= e^{tA}\psi_0+\int_{s=0}^{t}e^{(t-s)A}u(s)Be^{sA}\psi_0\mathrm{d}s \\
&\quad= W_0(t,u)\psi_0+W_1(t,u)\psi_0.
\end{aligned}$$

Assume the result is true for some integer p. Then, we write, for every t in $[0,T]$,

$$\begin{aligned}
&H_T\left(t\mapsto\sum_{j=0}^{p}W_j(t,0)\psi_0)(t)\right) \\
&\quad= e^{tA}\psi_0+\int_{s=0}^{t}e^{(t-s)A}u(s)B\sum_{j=0}^{p}W_j(t,0)\psi_0\mathrm{d}s \\
&\quad= W_0(t,u)\psi_0+\sum_{j=0}^{p}W_{j+1}(t,u)\psi_0 \\
&\quad= \sum_{j=0}^{p+1}W_j(t,u)\psi_0
\end{aligned}$$

and the result is true also at rank $p+1$. ∎

Proposition 16: For every u in $L^1_{loc}([0,+\infty),\mathbf{R})$, $T\geq 0$, and ψ_0 in X

$$\begin{array}{llll}
\Upsilon^u_{\cdot,0}\psi_0 & [0,+\infty) & \to & X \\
 & t & \mapsto & \sum_{p=0}^{\infty}W_p(t,u)\psi_0,
\end{array}$$

is the unique mild solution of (1).

Proof: Let $T>0$. From Proposition 13, the restriction of any mild solution of (1) to $[0,T]$ is a fixed point of H_T. From Proposition 14, there exists k in $\mathbf{N}$ large enough such that H_T^k is a contraction from $\mathcal{C}([0,T],X)$ to itself. Hence H_T^k admits one and only one fixed point in $\mathcal{C}([0,T],X)$. Since every fixed point of H_T is also a fixed point of H_T^k, the mapping H_T has at most one fixed point in $\mathcal{C}([0,T],X)$. Letting T vary from 0 to ∞, this means that (1) admits at most one mild solution.

From Proposition 11, for every t in $[0,T]$, the sum $\sum_{j=0}^{\infty}W_j(t,0)\psi_0)$ converges in X to a limit we note $\Upsilon^u_{t,0}\psi_0$. The aim of the rest of the proof is to show that $t\mapsto\Upsilon^u_{t,0}\psi_0$ is a mild solution of (1).

Let $T>0$. The convergence of the sum $\sum_{j=0}^{\infty}W_j(t,0)\psi_0$ being uniform with respect to t in $[0,T]$, the function $t\mapsto\Upsilon^u_{t,0}\psi_0$ is continuous. From Proposition 15, for every t in $[0,T]$, for every p in $\mathbf{N}$,

$$H_T\left(t\mapsto\sum_{j=0}^{p}W_j(t,u)\psi_0\right)=\left(t\mapsto\sum_{j=0}^{p+1}W_j(t,u)\psi_0\right).$$

As p tends to infinity, $\sum_{j=0}^{p}W_j(t,u)\psi_0$ tends to $\Upsilon^u_{t,0}\psi_0$. Since H_T is Lipschitz continuous (Proposition 15 with $k=1$), $H_T(t\mapsto\sum_{j=0}^{p}W_j(t,u)\psi_0)$ tends to $H_T(t\mapsto\Upsilon^u_{t,0}\psi_0)$ as p tends to infinity. In other words, the function $t\mapsto\sum_{j=0}^{\infty}W_j(t,0)\psi_0$ is a fixed point of H_T (the only one from what precedes). In conclusion, $t\mapsto\Upsilon^u_{t,0}\psi_0$ is the unique mild solution of (1). ∎

IV. PROOF OF THEOREM 2

We proceed now to the proof of Theorem 2. First of all, notice that, for every ψ_0 in X,

$$\begin{aligned}
&\mathcal{A}(\psi_0,L^1_{loc}([0,+\infty),\mathbf{R})) \\
&\qquad= \bigcup_{l\in\mathbf{N}}\bigcup_{m\in\mathbf{N}}\{\Upsilon^u_{t,0}\psi_0,\|u\|_{L^1}\leq l,0\leq t\leq m\},
\end{aligned}$$

and it is enough to prove that, for every l and m in $\mathbf{N}$, the set

$$\{\Upsilon^u_{t,0}\psi_0, \|u\|_{L^1} \leq l, 0 \leq t \leq m\}$$

is totally bounded.

Let $\varepsilon > 0$. From the convergence of the Dyson expansion (Proposition 16) and the bound on the operators W_j (Proposition 11), there exists a integer N_ε such that

$$\left\| \sum_{p \geq N_\varepsilon} W_p(t,u)\psi_0 \right\| \leq \frac{\varepsilon}{2}, \tag{8}$$

for every t in $[0,m]$ and every u such that $\|u\|_{L^1} \leq l$. For each $j = 1, \ldots, N_\varepsilon$ the sets $\mathcal{W}^{m,l}_j$, defined by (3), are totally bounded (Lemma 12), hence their sum

$$\sum_{j=0}^{N_\varepsilon} \mathcal{W}^{m,l}_j$$

is totally bounded as well (Proposition 8). Hence there exists a family $(x_i)_{1 \leq i \leq N_1}$ of points of $\sum_{j=0}^{N_\varepsilon} \mathcal{W}^{m,l}_j$ such that

$$\sum_{j=0}^{N_\varepsilon} \mathcal{W}^{m,l}_j \subset \bigcup_{i=1}^{N_1} B_X\left(x_i, \frac{\varepsilon}{2}\right). \tag{9}$$

Gathering (8) and (9), one gets

$$\sum_{j=0}^{\infty} \mathcal{W}^{m,l}_j \subset \bigcup_{i=1}^{N_1} B_X\left(x_i, \varepsilon\right),$$

which concludes the proof of Theorem 2.

V. ACKNOWLEDGMENTS

This work has been supported by the project DISQUO of the DEFI InFIniTI 2017 by CNRS and the QUACO project by French Agence Nationale de la Recherche, contract 17-CE40-0007-01.

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